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Symmetry breaking patterns and extended Morse theory

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Abstract. An indirect topological method is considered which constrains the possible breaking patterns of a Lie group symmetry for Higgs–Landau systems. The application of the method and its efficiency are discussed.

1. Introduction

The phenomenon of spontaneous symmetry breaking arises in many physical situations. That is where the symmetry of a system changes discontinuously with a continuous change of an external parameter of the system. A description of these symmetry changes is given, in Landau theory of phase transitions, by some polynomial $P(\phi)$ usually of degree four or less in the order parameter or Higgs field, ϕ, which is valued in some vector space V. The Higgs–Landau polynomial of the system, $P(\phi)$, is invariant under the linear action of a group G on V, is bounded below and has a local maximum at $\phi=0$. The actual symmetry of the system is given by the stability subgroup $H<G$ of the absolute minimum of $P(\phi)$. The symmetry is said to be spontaneously broken from G to H and H may change in a discontinuous way with a continuous change of the coefficients of $P(\phi)$.

The group G may be a discrete group, as in Michel and Mozrzymas (1978), or it may be a Lie group, as for the grand unified theories (GUT) of particle physics (George and Glashow 1974). (For reviews see O'Raifeartaigh (1979), Frampton *et al* (1980), Zee (1982).) In either case the problems encountered in minimising the Higgs–Landau polynomial may be formidable and it is of interest to develop methods to obtain the broken symmetry group H with less effort. For problems of symmetry breaking in crystallography, in which case G is discrete, Michel and Mozryzymas (1978) give a procedure for finding the broken symmetry *without explicitly minimising* $P(\phi)$. Their method is topological in nature and involves the use of the Morse inequalities as, in their case, $P(\phi)$ is a Morse function (i.e. the critical points of $P(\phi)$ are non-degenerate) (Bott 1982).

For models of GUT, such as the SU(5) model of Georgi and Glashow (1974), the group ($G=\mathrm{SU}(5)$) is a Lie group and, the stabiliser of the minimum of the Higgs potential $P(\phi)$, at least at the tree level, defines the broken symmetry to be e.g. $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. Using the Higgs mechanism, then, the masses for the vector gauge Bosons, the Fermions and the Higgs fields ϕ are obtained. When G is a Lie group, as for GUT, in order that the broken symmetry group H also be a Lie group the critical points of $P(\phi)$ must have some degeneracy. Thus $P(\phi)$ is not a Morse

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function and the method of Michel and Mozrzymas (1978) cannot now be applied directly. Our purpose in this paper is to suitably adapt the Michel–Mozrzymas strategy so as to be applicable to the case in which G is a Lie group and $P(\phi)$ is not a Morse function. As we shall see the appropriate generalisation involves the concept of an extended Morse function (Bott 1982).

A summary of the strategy for determining the possible candidate stabilisers of the minimum of $P(\phi)$ (without explicitly minimising $P(\phi)$) is as follows. First, write down all non-trivial stabilisers of the action of G on V and the corresponding G-orbits (for a review of the group theory involved see Michel (1980)); second, regarding $P(\phi)$ as an extended Morse function defined on the sphere S^D, $D = \dim V$, write down the extended Morse inequalities where the only critical manifolds allowed are G-orbits; third, write down any additional inequalities e.g. an upper bound on the number of critical manifolds which can occur, etc; finally, from these inequalities see what stabilisers are allowed for the minimum of $P(\phi)$. As is evident, the nature of the approach is topological and algebraic and we will only obtain information on local minima of $P(\phi)$. We will not be able to distinguish the absolute minimum from among these.

The remainder of the paper has been arranged as follows. In § 2 we give a precise formulation of the background to the problem of symmetry breaking being considered. In § 3 we review relevant points from extended Morse theory and apply it to the problem in hand. In § 4 we collect some results on roots of polynomial equations which will give additional inequalities to those of § 3. Section 5 contains a worked example illustrating the usefulness of the method and our conclusions. An appendix on the cohomology of homogeneous spaces is included.

2. On G-invariant polynomials

In the case of spontaneous symmetry breaking phenomena encountered in e.g. GUT we are presented with the following general mathematical structure. A compact, connected Lie group, G, acts on a real vector space V in a linear way with no non-zero fixed points. Under $g \in G$ the Higgs field $\phi \in V$ transforms to $\Delta(g)\phi \in V$ (Δ defines a real representation of G on V). For any point ϕ in V the set of all elements in G which leave ϕ fixed is called the stabiliser of ϕ and denoted G_ϕ and is a subgroup of G. Letting G act on a point ϕ generates a subset of V, $G \cdot \phi$, called the orbit of ϕ. All points in the same orbit have stabilisers in the same conjugacy class and, in fact, we can partition V into strata by use of the conjugacy classes of the stabilisers of its points. We let $(\cdot,\cdot)$ be an inner product on V which is G-invariant (i.e. $(\Delta(g)\phi_1, \Delta(g)\phi_2) = (\phi_1, \phi_2)$, for all $g \in G$, $\phi_1, \phi_2 \in V$) and we let $\|\phi\| = (\phi, \phi)^{1/2}$ denote the norm of ϕ.

The group G corresponds to the symmetry group of the system being considered. To obtain the true symmetry we must be given a Higgs–Landau polynomial $P(\phi)$ on V which we take to have the following properties:

(i) $P(\phi)$ is invariant under the action of G i.e. $P(\Delta(g)\phi) = P(\phi)$ for all $g \in G$, $\phi \in V$,

(ii) $P(\phi)$ has a local maximum at $\phi = 0$,

(iii) $P(\phi)$ is bounded below and $P(\phi) \to \infty$, as $\|\phi\| \to \infty$.

(Additional assumptions we can require on physical grounds are

(iv) $P(\phi)$ is of maximum degree 4,

(v) $P(\phi) = P(-\phi)$.)

The true symmetry group of the system has the same conjugacy class as that of the stabiliser of a point on the minimum orbit of $P(\phi)$ (note that if ϕ is a critical point of $P(\phi)$ then so is $\Delta(g)\phi$ for any $g \in G$).

Let $I_1^{(2)}(\phi), I_2^{(2)}(\phi), \ldots, I_{r_2}^{(2)}(\phi), \ldots, I_{r_k}^{(k)}(\phi)$ (not all values from 2 to k may occur in the superscript) be independent polynomial G-invariants, in terms of which any G-invariant function may be expressed (Michel 1980). Here the superscript denotes the degree of the invariant, i.e. $I_s^{(\alpha)}(\lambda\phi) = \lambda^\alpha I_s^{(\alpha)}(\phi)$. Then $P(\phi)$ can be written as an algebraic function of $I_1^{(2)}, \ldots, I_{r_k}^{(k)}$ (not necessarily as a polynomial function)

$$P(\phi) = \tilde{P}(I_1^{(2)}, \ldots, I_{r_k}^{(k)}). \tag{1a}$$

A typical form for $P(\phi)$, satisfying the assumptions (i)–(v), is

$$P(\phi) = \sum_i a_i^{(4)} I_i^{(4)} + \sum_{ij} a_{ij}^{(2,2)} I_i^{(2)} I_j^{(2)} + \sum_i a_i^{(2)} I_i^{(2)}, \tag{1b}$$

where the coefficients $a^{(\cdot)}$ are constants and are the physical input parameters of the system. Thus a full solution to the symmetry breaking problem would be given by finding the conjugacy class of the stabiliser of a point on the minimum orbit of $P(\phi)$ and its dependence on $\{a^{(\cdot)}\}$. This is an extremely difficult task, in general, and has been carried out for only special representations Δ (Frampton *et al* 1980, Kim 1983a) (for a recent status report see Kim (1983b)). The problem we will address here, as in Michel and Mozrzymas (1978), is a somewhat less difficult one. It consists in determining the allowed conjugacy classes of stabilisers for minima of $P(\phi)$ which can occur for some values of $\{a^{(\cdot)}\}$.

Before ending this section, let us note some properties of the invariants. First $(I_1^{(2)}, \ldots, I_{r_k}^{(k)})$ may vary independently in a region S of $\mathbb{R}^n$, $n = \Sigma_s r_s$ ($n \leq \dim V$), defined by n inequalities

$$\xi_j(I_1^{(2)}, \ldots, I_{r_k}^{(k)}) \geq 0, j = 1, \ldots, n. \tag{2}$$

The region of V where equation (2) holds as a strict inequality is the generic stratum and is open and dense in V (Michel 1980). The lower-dimensional strata are given by equation (2) with various combinations of the equality holding. Since knowing the stratum in which a point of V lies is the same as knowing the conjugacy class of its stability subgroup we may repose the spontaneous symmetry problem as a nonlinear programming problem, namely: minimise equation (1a) (or (1b)) with $(I_1^{(1)}, \ldots, I_{r_k}^{(k)})$ subject to equation (2). One consequence of this formulation of the problem in the case of equation (1b) is that since there $I(\phi)$ is linear in $I_1^{(4)}, \ldots, I_{r_4}^{(4)}$ the minimum must have $(I_1^{(2)}, \ldots, I_{r_k}^{(k)})$ on the boundary of S and outside the generic stratum.

3. Extended Morse inequalities

By virtue of assumption (iii) in § 2 we may regard $P(\phi)$ topologically as a C^∞ function from the sphere S^D ($D = \dim V$) to $\mathbb{R}$ (i.e. we regard V union ∞ as S^D). Moreover, $P(\phi)$ has a local maximum at $\phi = \infty$ (as well as $\phi = 0$). The set of local minima of $P(\phi)$ on V is contained in the set of critical points of $P(\phi)$

$$\nabla P(\phi) = 0. \tag{3}$$

Since $P(\phi)$ is G-invariant and since $\phi = 0$ and $\phi = \infty$ are the only fixed points of G

it follows that the Hessian of a critical point of $P(\phi)$ is degenerate i.e. $P(\phi)$ is not a Morse function. However, $P(\phi)$ may be taken to be an extended Morse function.

Let M be a compact connected manifold of dimension m and let f be a C^∞ real valued function on M. Then a connected submanifold N of M is said to be a non-degenerate critical manifold of M with respect to f if (Bott 1982):

(i) each point of N is a critical point of f,

(ii) the Hessian of f is non-degenerate in the normal direction to N.

This second condition means that in a neighbourhood of a point of N we can choose coordinates $(x^1, \dots, x^l, x^{l+1}, \dots, x^m)$ on M ($l = \dim N$) so that locally N is given by $x^i = 0$, $i = l+1, \dots, m$ and then we require

$$\det(\partial^2 f/\partial x^i\, \partial x^j)|_{x^{l+1}=\dots=x^m=0} \neq 0, \qquad i, j = l+1, \dots, m. \tag{4}$$

We say that f is a non-degenerate Morse function in the extended sense if all its connected critical sets are non-degenerate critical manifolds.

We denote the Poincaré polynomial of M with real coefficients by $\mathscr{P}_t(M)$ i.e.

$$\mathscr{P}_t(M) = \sum_{i=0}^{m} t^i b_i(M), \tag{5}$$

where $b_i(M)$ is the ith Betti number i.e. $b_i(M) = \dim H^i(M, \mathbf{R})$, $H^i(M, \mathbf{R})$ being the ith cohomology group of M with real coefficients. We denote $\mathscr{P}_t(N), \dots$ similarly. Let f be a non-degenerate Morse function in the extended sense. For any element N of $C(f)$, the set of connected critical manifolds of f on M, we let λ_N, called the index of N relative to f, denote the number of negative eigenvalues of the matrix in equation (4). Defining the Morse series of f over the real field of coefficients by

$$\mathscr{M}_t(f) = \sum_{N \in C(f)} t^{\lambda_N} \mathscr{P}_t(N), \tag{6}$$

we have the extended Morse inequalities:

Theorem.

$$\mathscr{M}_t(f) - \mathscr{P}_t(M) = (1+t) Q_t(f) \tag{7}$$

where $Q_t(f)$ is a polynomial in t of degree $m-1$ with non-negative coefficients.

When N is a minimum $\lambda_N = 0$, when N is a maximum $\lambda_N = \dim M - \dim N$ and otherwise N is a saddle.

For application purposes a more suitable form of equation (7) is desirable. We partition the set of critical manifolds $C(f)$ into classes with the same Betti numbers. We denote a typical element of each class by $N^{(\mu)}$ and let $d_{N^{(\mu)},j}$ be the number of critical manifolds in the same class as $N^{(\mu)}$ with index j, $j = 0, \dots, m$. The Morse function $\mathscr{M}_t(f)$ can now be written as $\mathscr{M}_t(f) = \sum_{i=0}^{m} c_i(f) t^i$ where

$$c_i(f) = \sum_{N^{(\mu)}} \sum_{i=0}^{\min(\dim N^{(\mu)}, j)} b_i(N^{(\mu)}) d_{N^{(\mu)}, j-i}. \tag{8}$$

The number of minimal (maximal) critical manifolds in the same class as $N^{(\mu)}$ is given by $d_{N^{(\mu)},0}$ $(d_{N^{(\mu)}, \dim M - \dim N^{(\mu)}})$. The Morse inequalities of equation (7) take the form

$$\sum_{i=0}^{l} (-1)^{l-i} c_i(f) \geq \sum_{i=0}^{l} (-1)^{l-i} b_i(M), \qquad 0 \leq l \leq m-1,$$

$$\sum_{i=0}^{m} (-1)^{m-i} c_i(f) = \sum_{i=0}^{m} (-1)^{m-i} b_i(M). \tag{9}$$

We now return to the Higgs–Landau polynomial $P(\phi)$. We regard $P(\phi)$ as an extended Morse function on S^D, $D = \dim V$ which has a local maximum at $\phi = 0$ and $\phi = \infty$. The only critical manifolds that we allow $P(\phi)$ to have are G-orbits. Thus to apply the Morse inequalities of equation (9) we need only know the cohomology groups $H^*(S^D, \mathbb{R})$ and $H^*(G\cdot\phi, \mathbb{R})$. The former cohomology is well known ($\mathscr{P}_t(S^D) = 1 + t^D$) and the latter may be computed in a straightforward way†. We give the details in the appendix.

4. Number of roots of a system of equations

The inequalities of § 3, on their own, are not very restrictive. They can become restrictive when combined with additional information. In Michel and Mozrzymas (1978) this additional information took the form of an upper bound on the total number of critical points. Thus we now try to get some bounds as the number of critical manifolds of $P(\phi)$.

We can express equations (3) for the critical points of $P(\phi)$ in terms of the invariants as follows. Write

$$(\nabla I_i^{(\alpha)}(\phi), \nabla P(\phi)) = 0, \qquad i = 1, \ldots, r_\alpha, \tag{10}$$

with $P(\phi)$ given by equation (1) and express equation (10) in terms of the invariants using

$$(\nabla I_i^{(\alpha)}(\phi), \nabla I_j^{(\beta)}(\phi)) = \sum_{\substack{\gamma_1\cdots\gamma_n \\ \Sigma\gamma = \alpha+\beta-2}} K_{ij}^{(\alpha\beta;\gamma_1\cdots\gamma_n)} I_1^{(2)\gamma_1} \ldots I_{r_k}^{(k)\gamma_n}, \tag{11}$$

where the coefficients $K^{(\cdot)}$ are numerical constants which can be determined directly. The resulting form for the equations for a critical point may be written

$$\sum_{\delta\in S_j} e_{j,\delta} z^\delta = 0, \qquad j = 1, \ldots, n, \tag{12}$$

where S_j is contained in $\mathbb{N}^n$, $\delta = (\delta_1, \ldots, \delta_n)$, $z^\delta = z_1^{\delta_1} \ldots z_n^{\delta_n}$ and $z = (I_1^{(2)}, \ldots, I_{r_k}^{(k)})$. In order to avoid the pseudo-Goldstone Boson problem‡ we will assume that only isolated roots exist to the critical point equations when written in the form (12) for the invariants.

Regarding z as a multi-component complex variable and equations (12) as complex equations we quote two theorems governing the number of roots to the system:

Theorem. (Bézout) (Semple and Roth 1949). If the degree of $\Sigma_{\delta\in S_j} e_{j,\delta} z^\delta$ is θ_j, $j = 1, \ldots, n$ then the number of roots of the system (12) including multiplicities is $\theta_1\theta_2 \ldots \theta_n$.

A more elaborate result given in Bernshtein (1975) is:

Theorem. If no component of z is zero, the number of roots of equations (12) is given by the mixed Minkowski volume $V_M(S_1, \ldots, S_n)$.

† It is possible to restate this section with the cohomology having coefficients in an arbitrary field. Thus the restriction to real coefficients can mean a loss in information but we make it on practical grounds as the cohomology of homogeneous spaces (i.e. of G-orbits) is difficult to obtain for coefficients other than for $\mathbb{R}$.
‡ That is, $P(\phi)$ should not be invariant under a symmetry group larger than G.

The value of $V_M(S_1, \ldots, S_n)$ is defined by

$$V_M(S_1, \ldots, S_n) = (-1)^{n-1} \sum_i V_n(S_i)$$
$$+(-1)^{n-2} \sum_{i<j} V_n(S_i + S_j) + \ldots + V_n(S_1 + \ldots + S_n), \tag{13}$$

where $V_n(S)$ denotes the Euclidean volume in $\mathbb{R}^n$ of the convex hull of the set S (the volume of the unit n-cube being 1) and where $S_1 + S_2 = \{q_1 + q_2 | q_i \in S_i\}$. In the latter theorem if one of the roots has a zero component, z_i say, we choose a different variable $z_1' = z_1 + a$ say, where z_1' does not have the value zero.

When $\boldsymbol{z}$ is real valued the numbers in the above two theorems give upper bounds to the number of possible roots of equation (12).

5. Application

We will now give, in detail, the strategy applying the material in §§ 3 and 4 for finding all allowed spontaneous symmetry breaking patterns. We will then consider the prototype example of $G = \mathrm{SU}(4)$, with the representation Δ being the adjoint representation of G, in order to test the method and discuss its efficiency.

As we have already seen, the conjugacy class of the stabiliser of a point ϕ of V is obtained from the stratum in which the orbit $G \cdot \phi$ lies (i.e. by knowing the orbit type). Thus we seek to find, within topological constraints, what are the allowed orbit types and their number for the critical manifolds of $P(\phi)$. In particular we are interested in the number allowed to be minimal orbits. We proceed as follows.

(a) We list all the possible G-orbit types (i.e. a typical orbit from each stratum), their Poincaré polynomials and the conjugacy classes of their stabilisers. The critical manifolds of $P(\phi)$ are all G-orbits by the assumption that equations (10) and (11) have only isolated roots. Thus the various types of critical manifolds are also G-orbit types. Let us, as in § 3, further partition the set of orbits (critical manifolds) into classes with the same Betti numbers and denote a typical element of each class by $N^{(\mu)}$ (μ labelling the class). The number of critical orbits in the same class as $N^{(\mu)}$ and with index j is again denoted by $d_{N^{(\mu)},j}$. This number is unknown except for a maximum at $\phi = 0$ and $\phi = \infty$ i.e. $d_{\{0\},D} = d_{\{\infty\},D} = 1$ ($D = \dim V$). In particular we look for the number of local minima of $P(\phi)$, $d_{N^{(\mu)},0}$, allowed.

(b) We compute $\{d_{N^{(\mu)},j}\}$ which satisfy the Morse inequalities

$$\sum_{j=0}^{l} (-1)^{l-j} c_j(P) \geqslant \sum_{j=0}^{l} (-1)^{l-j} b_j(S^D), \qquad 0 \leqslant l \leqslant D-1,$$
$$\sum_{j=0}^{D} (-1)^{D-j} c_j(P) = \sum_{j=0}^{D} (-1)^{D-j} b_j(S^D), \tag{14}$$

where $b_j(S^D) = \delta_{0,j} + \delta_{D,j}$ and

$$c_j(P) = \sum_{N^{(\mu)}} \sum_{l=0}^{\min(\dim N^{(\mu)}, j)} b_l(N^{(\mu)}) d_{N^{(\mu)},j-l}. \tag{15}$$

(c) Let T_p denote an upper bound on the number of critical manifolds, i.e. isolated roots of the critical equations (10) and (11), as given using § 4. (It is easy to compute a general estimate for T_p in terms of the degrees of $P(\phi)$ and $I_1^{(2)}(\phi), \ldots, I_{r_k}^{(k)}(\phi)$.

However we do not give one as it can usually be improved upon for any particular $P(\phi)$.) We can now compute the subset of $\{d_{N^{(\mu)},j}\}$ remaining from (b) allowed by the upper bound

$$\sum_{N^{(\mu)},j} d_{N^{(\mu)},j} \leq T_p + 1 \tag{16}$$

(the extra 1 coming from the maximum at $\phi = \infty$). Further restrictions on $\{d_{N^{(\mu)},j}\}$, such as not allowing generic critical orbits, may also be used if e.g. $P(\phi)$ is of the form $(1b)$.

By way of illustration we now consider the example: $G = \mathrm{SU}(4)$ with Δ being the adjoint representation ($\dim G = \dim V = 15$). There are three independent invariants $I^{(s)}(\phi)$, $s = 2, 3, 4$, and any non-trivial stabiliser is conjugate to one of the following groups: $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(3))$, $\mathrm{S}(\mathrm{U}(2)^2)$, $\mathrm{S}(\mathrm{U}(1)^2\times\mathrm{U}(2))$, $\mathrm{S}(\mathrm{U}(1)^4)$. Typical orbits from each stratum are then, respectively, $N^{(1)} = \mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(3))$, $N^{(2)} = \mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(2)^2)$, $N^{(3)} = \mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(1)^2\times\mathrm{U}(2))$, $N^{(4)} = \mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(1)^4)$ ($N^{(4)}$ being in the generic stratum) (Michel 1980). The Poincaré polynomials of $N^{(\mu)}$, $\mu = 1, 2, 3, 4$, are obtained from the formula ($k_i > 0, \Sigma k_i = l \leq h$) (Greub *et al* 1976)

$$\mathscr{P}_t\left(\frac{\mathrm{SU}(h)}{\mathrm{S}(\mathrm{U}(k_1)\times\ \ldots\ \times\mathrm{U}(k_q))}\right) = \frac{\pi_{p=1}^{l}(1-t^{2p})\,\pi_{p=l+1}^{n}(1+t^{2p-1})}{\pi_{i=1}^{q}\,\pi_{p=1}^{k_i}(1-t^{2p})}. \tag{17}$$

We have computed $\{d_{N^{(\mu)},j}\}$ satisfying (a), (b), (c) above. (The numerical work involved has been carried out using a Fortran program.) In the table we give the allowed minima, $d_{N^{(\mu)},0}$ and the number, #, of configurations of critical manifolds with those minima (the number in parentheses denotes the value, if different, obtained assuming no generic critical manifolds). We have allowed the upper bound T_P to have values $T_P \leq 5$ (there are no possibilities for $T_P \leq 3$).

From the table it is clear that the number of critical manifolds which can arise is highly sensitive to the value of the bound T_P. The most optimistic situation occurs when $T_P = 4$ as there is only one possible breaking pattern of SU(4) allowed, namely SU(4) breaks to $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(3))$. Even in the case $T_P = 5$ the information contained in the table is useful as it tells us, for instance, that there is only one possibility, (*), of having one minimal orbit of type $N^{(1)}$ and another of type $N^{(2)}$, together. Thus we

Table 1. We give the local minima numbers, $d_{N^{(\mu)},0}$ and the number, #, of configurations of critical manifolds allowed by (a), (b), (c) with these minima, for a bound T_P, where $G = \mathrm{SU}(4)$ and Δ = adjoint representation. The value in parentheses denotes the number of non-generic critical orbits (if different). We have not displayed the non-minimal critical orbit number, $d_{N^{(\mu)},j}$, $j \neq 0$, for the sake of convenience. (E.g. for (*) we have $d_{N^{(3)},1} = d_{N^{(1)},8} = 1$, $d_{N^{(\mu)},j} = 0$ otherwise.)

T_P	$d_{N^{(1)},0}$	$d_{N^{(2)},0}$	$d_{N^{(3)},0}$	$d_{N^{(4)},0}$	#
4	1	0	0	0	1
5	1	0	0	0	10
	2	0	0	0	1
	0	1	0	0	22 (18)
	1	1	0	0	$1^{(*)}$
	0	0	1	0	5
	0	0	0	1	4 (0)

could have a first-order phase transition with the symmetry pattern changing from S(U(1) × U(2)) to S(U(2)^2).

We have mentioned previously that the use of real cohomology, alone, may result in a loss of information. By this we mean that some types of orbits may be identified. Hence, the corresponding stability subgroups may be identified. (Two types of orbits will be identified if their integer-cohomologies differ only by torsion.) Further work can be done to resolve this identification e.g. by making use of integer-mod 2-cohomology. However, as regards examples from physics, we can expect that the information obtained using real-cohomology alone is substantial and, in particular, for the SU(4) example above the integer-cohomology of each $N^{(\mu)}$ is torsion free (i.e. there is no information loss).

Appendix. The cohomology of homogeneous spaces

We need to know the Poincaré polynomials, i.e. the cohomology, of the orbits of the action of G on V. For cohomology with real coefficients this is a straightforward task. We shall now give a theorem which enables the computation of this cohomology. Tables of the Poincaré polynomials of standard cases may be found in Greub *et al* (1976 pp 492–7).

First we note that we may identify an orbit $G \cdot \phi$ with the coset space or homogeneous space G/G_ϕ. Thus our task is to consider the cohomology of the homogeneous spaces G/H where H is a subgroup of G. Letting G be a compact connected Lie group and H a closed subgroup of G, then G/H is a topological manifold. Let $\mathscr{G}$ denote the Lie algebra of G and $\mathscr{H}$ the subalgebra of $\mathscr{G}$ corresponding to H. Let Ad denote the adjoint action of G on $\mathscr{G}$ and let $\langle\, ,\rangle$ denote the Ad-invariant inner product on $\mathscr{G}$ (the Cartan-Killing form). Then by $\mathscr{H}^\perp$ we mean the orthogonal complement to $\mathscr{H}$ in $\mathscr{G}$ with respect to $\langle\, ,\rangle$. We denote by $\Omega^l(W)$ the vector space of l-multi-linear skew symmetric real-valued maps on $W(=\mathscr{G}, \mathscr{H}, \mathscr{H}^\perp)$ and let Ad* be the action on $\Omega^l(\mathscr{G})$ induced by Ad. We let $\Omega^l(\mathscr{H}^\perp)^H$ be the vector space of all w in $\Omega^l(\mathscr{H}^\perp)$ such that $\mathrm{Ad}(h)^* w = w$, for all h in H and we define d by

$$\begin{aligned} &\mathrm{d}w(X_1, \ldots, X_{l+1}) \\ &\quad = \sum_{i<j} (-1)^{i+j} w(\mathscr{H}^\perp \text{ component of } [X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{l+1}), \end{aligned} \tag{A1}$$

where $X_1, \ldots, X_{l+1}$ are in $\mathscr{H}^\perp$, w is in $\Omega^l(\mathscr{H}^\perp)$ and $[\cdot\,,\cdot]$ denotes the Lie product on $\mathscr{G}$ ($\hat{}$ means that the entry is omitted). We can compute the cohomology of G/H in terms of elementary algebraic properties of G, H and the imbedding of H in G with the following isomorphism (Spivac 1975).

Theorem.

$$H^l(G/H) \cong \frac{\mathrm{Ker}\ d\colon \Omega^l(\mathscr{H}^\perp)^H \to \Omega^{l+1}(\mathscr{H}^\perp)^H}{d(\Omega^{l-1}(\mathscr{H}^\perp)^H)}. \tag{A2}$$

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